Published by Institute of Physics Publishing for SISSA

Received: *October 31, 2007*
Accepted: *November 07, 2007*
Published: *November 20, 2007*

Toric Kähler metrics and AdS_5 in ring-like co-ordinates

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Abstract: Stationary, supersymmetric supergravity solutions in five dimensions have Kähler metrics on the four-manifold orthogonal to the orbits of a time-like Killing vector. We show that an explicit class of toric Kähler metrics provide a unified framework in which to describe both the asymptotically flat and asymptotically AdS solutions. The Darboux co-ordinates used for the local description turn out to be "ring-like." We conclude with an ansatz for studying the existence of supersymmetric black rings in AdS.

Keywords: Black Holes in String Theory, Black Holes.

Contents

1. Introduction and summary

Black holes can have non-spherical horizons in more than four spacetime dimensions. In five dimensions, one encounters the first example of a non-spherical horizon, $S^2 \times S^1$, a black ring.[1] Emparan and Reall constructed the first black ring solution as a solution to vacuum gravity [1]. Solutions in five dimensions can have two independent rotation parameters, call them J_ψ and J_ϕ. The Emparan-Reall ring rotates only along the direction of the ring, $J_\phi = 0$. Subsequently, black-ring solutions with $J_\psi = 0$ but $J_\phi \neq 0$ were discovered [2], again in vacuum gravity. A black ring solution to Einstein-Maxwell theory was constructed in [3].

Asymptotically flat supersymmetric black rings were constructed first in Elvang et al [5], though they had been conjectured to exist earlier in [4]. We will refer to this

[1] For a review, see [9].

solution as the *flat susic ring.* The flat susic ring has both the rotation parameters turned on, $J_\psi \neq 0, J_\phi \neq 0$. This solution fits into the classification scheme for supersymmetric solutions to five dimensional minimal supergravity, discovered by Gauntlett et. al. [6]. According to this classification, there are two classes of solutions, corresponding to whether the Killing spinor bilinear $\bar{\epsilon}\,\gamma^\mu\,\epsilon$, which is always a Killing vector, is time-like or null. In this paper we will only be interested in the time-like class; choosing the time co-ordinate, τ, along the orbits of the time-like Killing vector, the metric takes the following stationary form:

$$ds^2 = -f^2\,(d\tau + \omega)^2 + \frac{1}{f}\,ds^2_{M_4} \tag{1.1}$$

where f is a function, ω is a one-form and $ds^2_{M_4}$ is a four-metric on the space transverse to the orbits of the Killing vector. M_4 will be referred to as the *base-space.* It was shown in [6] that supersymmetry requires f to be a function on M_4, ω to be a one-form on M_4 and that the base-space be a hyper-Kähler manifold. These quantities must also satisfy the following p.d.e's.

$$d\,G^+ = 0, \qquad \Delta\,(1/f) = \frac{4}{9}\,|G^+|^2, \tag{1.2}$$

where

$$G^\pm = \frac{f}{2}\,(d\,\omega\ \pm *\,d\,\omega) \tag{1.3}$$

and Δ is the Laplacian for functions on the base-space. For the susic flat ring solution of [5], the base-space is just flat-space; however, Euclidean coordinates are not so useful here and one writes the flat-space metric in special co-ordinates that will be referred to as *C-metric* co-ordinates. In these coordinates, the above equations become simple to solve [12]. There are other supersymmetric black ring solutions which involve more non-trivial hyper-Kähler base-spaces: the Gauntlett-Gutowski rings [7, 8] that have a Gibbons-Hawking base-space.

An interesting question is the existence of supersymmetric black ring solutions in AdS space, i.e. ring solutions to gauged supergravity theories. The theory of supersymmetric solutions to gauged supergravity was first worked out by Gauntlett and Gutowski in [10], henceforth referred to as the Gauntlett-Gutowski theory, for the minimal case and subsequently for the $\mathrm{U}(1)^3$ case in [17]. Again there is a time-like class of solutions, whose metric takes the form (1.1) and as in the asymptotically flat case, f is a function on the base-space and ω is a one-form on the base-space. But now supersymmetry requires, both in the minimal and $\mathrm{U}(1)^3$ cases, that the base-space M_4 be a Kähler manifold.

We will import a class of $\mathrm{U}(1)^2$ invariant Kähler metrics from the mathematics literature [11]. The metrics are given in Darboux or symplectic co-ordinates; in these co-ordinates *all the metrics in the class have the same Kähler form*, whilst the complex structures can be different. We find in this class two important metrics: 1. the base-space metric for the susic flat ring and 2. the base-space metric for AdS_5, i.e. the Bergmann metric. The parameter that deforms the first to the second is precisely the cosmological constant.

1. For the flat-space metric, we find that the Darboux co-ordinates are "ring-like": the co-ordinate ranges of the Darboux co-ordinates in the co-ordinate directions orthogonal to the $\mathrm{U}(1)^2$ directions are identical to the corresponding C-metric co-ordinates

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and the flat-space metric in Darboux co-ordinates is isometric to the flat-space metric in C-metric co-ordinates. We give the explicit co-ordinate transformations.

2. We show that a certain metric in the class of metrics [11] is the Bergmann metric. The co-ordinate region in which this form of the Bergmann metric is defined is exactly the same as the co-ordinate region in which the flat-space metric is defined. We thus obtain the Bergmann metric and hence AdS_5 in ring-like co-ordinates. We give the explicit co-ordinate transformation between the polar co-ordinates in which the Bergmann metric is usually expressed and the "ring-like" Darboux co-ordinates.

Since in the Darboux co-ordinates, the Kähler form takes such a simple form, the underlying Kähler structure of susic flat solutions becomes prominent when we express them in these co-ordinates. We find that the one-form ω of the susic flat ring takes the following simple form:

$$\omega = \rho \; d\psi + k \; \Omega^{(1)}, \tag{1.4}$$

where $\Omega^{(1)}$ is a one-form whose exterior derivative is the Kähler form, ψ is the co-ordinate along the ring and ρ, k are functions. Apart from the susic flat ring, there are other $\mathrm{U}(1)^2$ invariant asymptotically flat supersymmetric solutions known. These describe a supersymmetric black hole placed in the centre of a susic flat ring [7, 8, 12, 13]. We describe these solutions in ring-like coordinates and show that the one-form ω also takes the form (1.4).

Recent work [24] suggests that supersymmetric black rings do not exist in AdS_5. Our ansatz provides a framework for addressing this interesting question further.

This paper is organized as follows. In section two, we reproduce all the necessary definitions, theorems from the math papers that we will need for our work. In section three, we consider flat-space in Darboux co-ordinates, describe in what sense they are "ring-like" and describe the relation between Darboux and C-metric coordinates. In section four, we elaborate on the properties of a certain $\mathrm{U}(1)^2$ invariant metric that allows us to conclude that it is the Bergmann metric, and obtain AdS_5 in "ring-like" co-ordinates. In section five, we co-ordinate transform various flat supersymmetric solutions from C-metric co-ordinates to the Darboux co-ordinates to find the form (1.4) for the one-form ω. In section six, we make Ansätze for the metric and the one-form of the AdS ring and conclude with directions for future work.

2. Toric Kähler metrics in Darboux co-ordinates

The expectation that a supersymmetric AdS ring should have a $\mathrm{U}(1)^2$ isometry, similar to the susic flat ring, leads us to look for Kähler metrics preserving a $\mathrm{U}(1)^2$ isometry. Such metrics have appeared before in both the math and physics literature and are referred to as *toric Kähler* metrics. More precisely, a toric Kähler metric is a Kähler metric admitting commuting holomorphic Killing vector fields which are independent. There exists a description of such metrics in local co-ordinates in the symplectic geometry literature. One thinks of the Kähler form as a symplectic form; in symplectic geometry, one uses local co-ordinates, called symplectic or Darboux co-ordinates, in which the symplectic form takes a standard form while the metric and the complex structure are described by non-trivial tensors. The study of Kähler metrics in symplectic co-ordinates is attributed to the mathematicians, Guillemin [14] and Abreu [15]. Symplectic co-ordinates also appear naturally in the context of the gauged linear sigma model [16]. Here, we reproduce the relevant proposition from [11].

Proposition 1. *Let G_{ij} be a positive definite 2×2 symmetric matrix of functions of 2-variables x_1 and x_2 with inverse G^{ij}. Then the metric*

$$\sum_{i,j} \left(G_{ij} dx_i dx_j + G^{ij} dt_i dt_j \right) \tag{2.1}$$

is almost-Kähler with Kähler form

$$\Omega = dx_1 \wedge dt_1 + dx_2 \wedge dt_2 \tag{2.2}$$

and has independent hamiltonian Killing vector fields $\partial/\partial t_1$, $\partial/\partial t_2$ with Poisson-commuting momentum maps x_1 and x_2. Any almost-Kähler structure with such a pair of Killing vector fields is of this form (where the t_i are locally defined up to an additive constant), and is Kähler if and only if G_{ij} is the Hessian of a function of x_1 and x_2.

The function of two variables is known as the *symplectic potential*. The Kähler metric that appears in the above proposition is still quite generic, being specified by an arbitrary function of two variables. Note that the two-form Ω takes a simple form in symplectic co-ordinates while the metric (and complex structure) are non-trivial. The authors of [11] define a sub-class of toric Kähler metrics viz. *ortho-toric Kähler* metrics.

Definition 2. *A Kähler metric is* ortho-toric *if it admits two independent hamiltonian Killing vector fields with Poisson-commuting moment maps $(\xi + \eta)$ and $\xi\,\eta$ such that $d\xi$ and $d\eta$ are orthogonal.*

In other words, an ortho-toric Kähler metric is a toric Kähler metric, which when expressed in the ξ-η co-ordinates does not contain cross-terms ($g^{\,\xi\eta} = 0$.)

$$x_1 = \xi + \eta, \qquad x_2 = \xi\,\eta. \tag{2.3}$$

One of the virtues of the ξ-η co-ordinates is that there is a symmetry under the exchange $\xi \leftrightarrow \eta$, which simplifies computations and allows for compact expressions. This feature is shared by the x-y part of the C-metric coordinates, about which we will elaborate later. We will henceforth refer to the $\{\xi, t, \eta, z\}$ co-ordinates as the *Darboux coordinates*, even if that term strictly should mean the $\{x_1, t, x_2, z\}$ co-ordinates. We will now quote the following proposition from [14], which provides a local form of ortho-toric Kähler metrics in the ξ-η coordinates.

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Proposition 3. *The almost-Hermitian structure* (g, J, Ω) *defined by*

$$g = (\xi - \eta)\left(\frac{d\xi^2}{F(\xi)} - \frac{d\eta^2}{G(\eta)}\right) + \frac{1}{\xi - \eta}\left(F(\xi)(dt + \eta\, dz)^2 - G(\eta)(dt + \xi\, dz)^2\right), \quad (2.4)$$

$$Jd\xi = \frac{F(\xi)}{\xi - \eta}(dt + \eta\, dz), \qquad Jdt = -\frac{\xi\, d\xi}{F(\xi)} - \frac{\eta\, d\eta}{G(\eta)},$$

$$Jd\eta = \frac{G(\eta)}{\eta - \xi}(dt + \xi\, dz), \qquad Jdz = \frac{d\xi}{F(\xi)} + \frac{d\eta}{G(\eta)}, \quad (2.5)$$

$$\Omega = d(\xi + \eta) \wedge dt + d(\xi\, \eta) \wedge dz \quad (2.6)$$

is an ortho-toric Kähler structure for any functions F, G *of one variable. Every ortho-toric Kähler structure is of this form, where* t, z *are locally defined up to an additive constant.*

The following one-form $\Omega^{(1)}$ whose exterior derivative gives the Kähler form (2.6) will appear later.

$$d\,\Omega^{(1)} = \Omega, \qquad \Omega^{(1)} \equiv (\xi + \eta)\, dt + (1 + \xi\eta)\, dz \quad (2.7)$$

It is worth re-emphasizing that all the Kähler metrics in (2.4) and (2.1) have the same Ω and $\Omega^{(1)}$ as given in (2.6) and (2.7).

We will refer to the metrics (2.4) as the ACG metrics. The authors of [11] derive their motivation to consider co-ordinates (2.3) and to focus on the ortho-toric sub-class of toric Kähler metrics from their study of what are called *weakly self-dual* Kähler metrics. A weakly self-dual Kähler metric is a Kähler metric whose anti-self-dual part of the Weyl tensor is harmonic, (after having fixed the underlying orientation to make the Kähler form self-dual.) A weakly self-dual Kähler metric has many important properties, one of which is that two particular functions built out of the metric are Poisson-commuting holomorphic potentials (a holomorphic potential is the moment map of a hamiltonian Killing vector field.) Poisson-commutation ensures that the corresponding hamiltonian Killing vector fields commute, though they need not be independent. The two functions in question are the normalized Ricci scalar,[2] s, and the pfaffian of the normalized Ricci form,[3] p. Furthermore, when expressed in coordinates related to the holomorphic potentials,

$$s = \xi + \eta, \qquad p = \xi\, \eta, \quad (2.8)$$

it turns out that a weakly self-dual Kähler metric does not contain cross-terms, i.e. $g^{\xi\eta} = 0$.

One can consider a ortho-toric weakly-self dual Kähler metric, i.e. a weakly self-dual Kähler metric for which the two Killing vector fields (whose moment maps are s and p) are independent and commuting. From the fact that they are ortho-toric, they should take the form (2.4) and the weak self-duality condition should pick out specific forms for $F(\xi)$ and $G(\eta)$. It turns out that for the following choices of $F(\xi)$ and $G(\eta)$,

$$F(x) = k\, x^4 + l\, x^3 + A\, x^2 + B\, x + C_1, \qquad G(x) = k\, x^4 + l\, x^3 + A\, x^2 + B\, x + C_2, \quad (2.9)$$

[2]This turns out to be $\frac{1}{6}$ of the Ricci scalar. The numerical factor is important while defining the co-ordinate transformation (2.8).

[3]The normalized Ricci form is the two-form $\frac{1}{2}\rho_0 + \frac{s}{4}\Omega$, where ρ_0 is the traceless Ricci form; and the pfaffian of a two-form ψ, pf(ψ), is the function that multiplies the volume form to give $\psi \wedge \psi$, i.e. $\psi \wedge \psi = \text{pf}(\psi)\, vol$.

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where k, l, A, B, C_1, C_2 are constants, the ortho-toric metric (2.4) is weakly self-dual [11].

The two cases of interest to us are when we have a quadratic polynomial for which the weakly self-dual metric is just flat space and when we have a cubic polynomial for which the weakly self-dual metric is a Kähler Einstein space. The former is the base-space for the full solution of a supersymmetric flat ring (more on this in section three) and the latter is the base-space for AdS_5 (more on this in section four). Therefore we see that these coordinates will be useful in the description of several known, important examples.

3. Flat space metric in Darboux co-ordinates

In this section, we will first gather various facts about the C-metric co-ordinates $\{x, \phi, y, \psi\}$, which will be relevant for the discussion on and comparison with the flat space metric in Darboux co-ordinates in the later subsection.

3.1 Flat space metric in C-metric co-ordinates

Following is the flat-space metric in C-metric co-ordinates,

$$ds^2 = \frac{1}{(x-y)^2}\left[\frac{dx^2}{1-x^2} + (1-x^2)d\phi^2 + \frac{dy^2}{y^2-1} + (y^2-1)d\psi^2\right]. \tag{3.1}$$

The co-ordinate ranges for the x and y co-ordinates can be inferred from (3.1) by requiring that the metric be positive definite:

$$-1 \le x \le 1, \qquad -\infty < y \le -1. \tag{3.2}$$

The metric (3.1) is thus defined in the region of the $x - y$ plane, shown in figure 1. The following nice physical interpretation of the C-metric co-ordinates was given in [9]. In flat space, given in the usual polar co-ordinates,

$$ds^2 = dr_1^2 + r_1^2 d\phi^2 + dr_2^2 + r_2^2 d\psi^2, \tag{3.3}$$

a circular string of unit radius stretched along the ψ direction, i.e. at $r_1 = 0, r_2 = 1$, acting as an electric source for the three-form field strength $H = dB$, produces a field with only the following non-zero two-form potential:

$$B_{t\psi} = -\frac{1}{2}\left(1 - \frac{1 + r_1^2 + r_2^2}{\Sigma}\right), \tag{3.4}$$

where

$$\Sigma = \sqrt{(1 + r_1^2 + r_2^2)^2 - 4\,r_2^2} \tag{3.5}$$

and the two-form field strength dual to H has a one-form potential ($*H = F = dA$) whose only non-zero component is

$$A_\phi = -\frac{1}{2}\left(1 + \frac{1 - r_1^2 - r_2^2}{\Sigma}\right). \tag{3.6}$$

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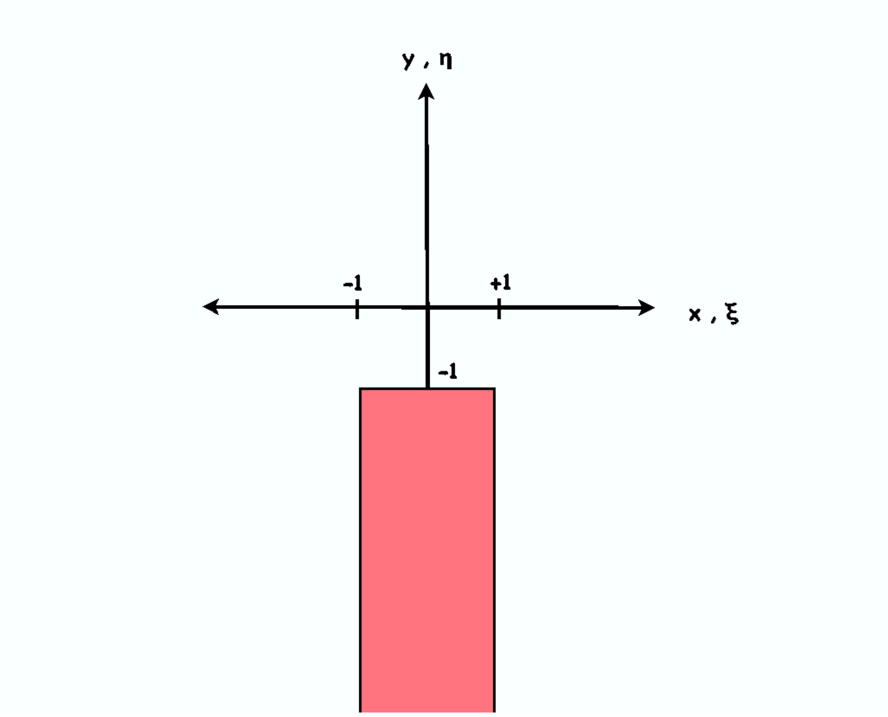

Figure 1: The region in the x-y and ξ-η planes, where the metrics (3.1), (3.10) and (4.2) are defined. Thus the supersymmetric flat ring and AdS_5 are defined in this co-ordinate region.

If one were to define new co-ordinates

$$x = \frac{1 - r_1^2 - r_2^2}{\Sigma}, \qquad y = -\frac{1 + r_1^2 + r_2^2}{\Sigma}, \tag{3.7}$$

then constant A_ϕ surfaces would be constant x surfaces and constant $B_{t\psi}$ surfaces would be constant y surfaces. These new co-ordinates $\{x, \phi, y, \psi\}$ are nothing but the C-metric co-ordinates. The metrics (3.3) and (3.1) are isometric which can be explicitly verified with (3.7) and it's inverse:

$$r_1 = \frac{\sqrt{1 - x^2}}{x - y}, \qquad r_2 = \frac{\sqrt{y^2 - 1}}{x - y}. \tag{3.8}$$

We thus see that the C-metric co-ordinates are "ring co-ordinates," adapted to describe the fields set up by a charged ring source.

3.2 ACG-quadratic metric

Choosing the same quadratic polynomial for $F(\xi)$ and $G(\eta)$ in the most general ortho-toric metric given by (2.4),

$$F(z) \;=\; G(z) \;=\; 1 - z^2, \tag{3.9}$$

we get the following flat metric on R^4, which we will refer to as the *ACG-quadratic* metric:

$$ds^2 = (\xi - \eta)\left[\frac{d\xi^2}{1 - \xi^2} + \frac{d\eta^2}{\eta^2 - 1}\right] + \frac{1}{\xi - \eta}\left[(1 - \xi^2)(dt + \eta\, dz)^2 + (\eta^2 - 1)(dt + \xi\, dz)^2\right]. \tag{3.10}$$

Requiring that the metric (3.10) be positive definite constrains the ranges of the ξ-η co-ordinates to:[4]

$$-1 \le \xi \le 1, \qquad -\infty < \eta \le -1, \tag{3.11}$$

which is exactly the same region in the $\xi - \eta$ plane as the one occupied by the C-metric co-ordinates, see figure 1. We can work out the explicit co-ordinate transformations between the polar $\{r_1, \phi, r_2, \psi\}$ co-ordinates and the $\{\xi, t, \eta, z\}$ co-ordinates.

$$\begin{aligned} r_1 &= \sqrt{-2(1+\xi\,\eta+\xi+\eta)}, & \phi &= \frac{t+z}{2}, \\ r_2 &= \sqrt{2(1+\xi\,\eta-\xi-\eta)}, & \psi &= \frac{t-z}{2}. \end{aligned} \tag{3.12}$$

The inverse of the above are:

$$\begin{aligned} \xi &= \frac{\Xi - r_1^2 - r_2^2}{8}, & t &= \phi + \psi, \\ \eta &= -\frac{\Xi + r_1^2 + r_2^2}{8}, & z &= \phi - \psi. \end{aligned} \tag{3.13}$$

where

$$\Xi = \sqrt{(8 + r_1^2 + r_2^2)^2 - 32\; r_2^2}. \tag{3.14}$$

3.3 C-metric co-ordinates $\leftrightarrows$ Darboux co-ordinates

From (3.11) and (3.2), it is clear that both the metrics (3.1) and (3.10) are defined in the same region (figure 1) of the x-y/ξ-η plane. Furthermore, the two metrics (3.1) and (3.10) are isometric to each other, which can be checked using the following explicit co-ordinate transformations between the $\{\xi, t, \eta, z\}$ co-ordinates and the $\{x, \phi, y, \psi\}$ co-ordinates, which can be worked out from (3.8) and (3.12):

$$\begin{aligned} x &= \frac{4(\xi+\eta)+1}{\Lambda}, & \phi &= \frac{t+z}{2} \\ y &= \frac{4(\xi+\eta)-1}{\Lambda}, & \psi &= \frac{t-z}{2}, \end{aligned} \tag{3.15}$$

where

$$\Lambda = \sqrt{1 - 8(1+\xi\,\eta) + 16(\xi+\eta)^2}. \tag{3.16}$$

The inverse of the above are:

$$\begin{aligned} \xi &= \frac{x+y+\Upsilon}{8\,(x-y)}, & t &= \phi+\psi, \\ \eta &= \frac{x+y-\Upsilon}{8\,(x-y)}, & z &= \phi-\psi, \end{aligned} \tag{3.17}$$

where

$$\Upsilon = \sqrt{4 + 28\,(1 - x\,y) + 7\,(x-y)^2}. \tag{3.18}$$

[4] We have made the choice $\xi \ge \eta$; the other choice $\xi \le \eta$ just interchanges ξ and η.

We will need the above in section five to co-ordinate transform various asymptotically flat solutions known in C-metric co-ordinates to the Darboux co-ordinates.

Both the C-metric and the Darboux co-ordinates share an exchange symmetry $x \leftrightarrow y$, $\xi \leftrightarrow \eta$, which simplifies computations and allows for compact expressions. We want to think of the Darboux co-ordinates as "ring-like" co-ordinates because they are so closely related to the C-metric "ring co-ordinates." A compelling reason to consider flat space in these Darboux co-ordinates is that we can simply deform the metric and the complex structure while keeping the Kähler form fixed to arrive at the base-space of AdS_5, as we will show in the next section.

4. AdS_5 in ring-like co-ordinates

4.1 The ACG-cubic Metric

Choosing the same cubic polynomial for $F(\xi)$ and $G(\eta)$ in the most general ortho-toric metric given by (2.4),

$$F(z) \;=\; G(z) \;=\; (1 - z^2)\,(1 + a\,p - z\,p), \tag{4.1}$$

with $a > (p-1)/p$ and $p > 0$ real constants, we get the following Kähler metric which we will refer to as the *ACG-cubic* metric:

$$\begin{aligned} ds^2_{\mathrm{ACG-cubic}} = (\xi - \eta)\left[\frac{d\xi^2}{(1-\xi^2)\,(1+a\,p-\xi\,p)} - \frac{d\eta^2}{(1-\eta^2)\,(1+a\,p-\eta\,p)}\right] \\ + \frac{1}{\xi-\eta}\Big[(1-\xi^2)\,(1+a\,p-\xi\,p)\,(dt+\eta\,dz)^2 \\ -(1-\eta^2)\,(1+a\,p-\eta\,p)\,(dt+\xi\,dz)^2\Big]. \end{aligned} \tag{4.2}$$

We find that for the co-ordinate ranges (3.11), the ACG-cubic metric is positive definite. Thus the ACG-cubic metric is defined in the same region of the $\xi - \eta$ plane as the ACG-quadratic metric (figure 1). Furthermore, setting $p \to 0$ gives us back the ACG-quadratic metric. Some properties of the ACG-cubic metric are:

(i) The ACG-cubic metric is a Kähler-Einsten metric. When one chooses a cubic polynomial as above, we are assured (by a consequence of propositions 4 and 11 of [11]) to get a Kähler-Einstein metric. One can compute and verify that the ACG-cubic metric is a Kähler-Einstein metric with $-6\,p$ for it's Ricci scalar,

$$R_{\mu\nu} = -\frac{3\,p}{2} g_{\mu\nu}. \tag{4.3}$$

(ii) The ACG-cubic metric has *constant holomorphic sectional curvature.* A Kähler metric with a constant holomorphic sectional curvature is the Kähler geometry equivalent of a maximally symmetric metric in Riemannian geometry. The Riemann tensor can be decomposed into three pieces: the fully traceless Weyl tensor (W), a partially

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traceless part that can be written in terms of the metric and the traceless Ricci tensor (call it P) and a trace that involves only the Ricci scalar (call it S). A metric with *constant sectional curvature* corresponds to having $W = P = 0$, which leads to the usual expression for a maximally symmetric metric:

$$R_{ijkl} = S_{ijkl} = \frac{R}{d(d-1)} \left(g_{ik}\, g_{jl} - g_{il}\, g_{jk}\right) . \tag{4.4}$$

In Kähler geometry, employing the extra symmetries of the Riemann tensor, one can similarly decompose again into three pieces (section 2.63 of [20]): a fully traceless part (called B_0 in [20]), a partially traceless part that can be written in terms of the Kähler form and the traceless Ricci form (call it P') and a piece that involves only the scalar curvature (call it S'.) A metric with constant holomorphic sectional curvature corresponds to having $B_0 = P' = 0$, which leads to the following expression for the Riemann tensor:

$$R_{ijkl} = S'_{ijkl} = \frac{R}{4\frac{d}{2}(\frac{d}{2}+1)} \left(g_{ik}\, g_{jl} - g_{il}\, g_{jk} + 2\,\Omega_{ij}\,\Omega_{kl} - \Omega_{il}\,\Omega_{jk} + \Omega_{jl}\,\Omega_{ik}\right) . \tag{4.5}$$

One can see that a metric with constant holomorphic sectional curvature is not maximally symmetric. The ACG-cubic metric (4.2) satisfies equation (4.5), which we can verify explicitly.[5]

(iii) The ACG-cubic metric is holomorphically isometric to the Bergmann metric. The Bergmann metric is a Kähler metric defined for the unit ball $\mathbf{B}^4 \subset \mathbf{C}^2$ (section 3.2 of [10]) in complex co-ordinates

$$z^1 = r\,\cos\frac{\theta}{2}\; e^{\frac{i(\phi+\psi)}{2}}, \qquad z^2 = r\,\sin\frac{\theta}{2}\; e^{\frac{i(\phi-\psi)}{2}}. \tag{4.6}$$

using the Kähler potential:

$$K = -\frac{2}{p}\log(1 - |z^1|^2 - |z^2|^2), \tag{4.7}$$

where the real co-ordinates in (4.6) are the radius r and the Euler angles. The explict form of the metric is:

$$ds^2_{\text{Bergmann}} = \frac{4}{p}\left[\frac{dr^2}{(1-r^2)^2} + \frac{r^2}{4(1-r^2)}(d\theta^2 + \sin\theta^2 d\phi^2) + \frac{r^2}{4(1-r^2)^2}(d\psi + \cos\theta\, d\phi)^2\right] \tag{4.8}$$

The Bergmann metric has a constant holomorphic sectional curvature. There is a theorem in Kähler geometry that says that *any two simply-connected complete Kähler manifolds with constant holomorphic sectional curvature are holomorphically isometric to each other* (see for example [21], page 179.) We can invoke this theorem to conclude that the ACG-cubic metric is holomorphically isometric to the Bergmann metric.

[5] We were informed by Harvey Reall that he and his collaborators were able to arrive at a closely related version of the ACG-cubic metric by imposing constant holomorphic sectional curvature on a certain class of Plebanski-Demianski metrics.

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(iv) The ACG-cubic metric has a constant Kretschmann scalar, i.e.

$$R_{ijkl}R^{ijkl}|_{\mathrm{ACG-cubic}} = 12\,p^2. \tag{4.9}$$

This is an indication that the metric is a homogeneous metric.[6] Given that the Bergmann metric is known to be a homogeneous metric, it being the metric on the coset space $\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)}$, this adds further weight to the conclusion that the ACG-cubic metric and the Bergmann metric are holomorphically isometric to each other.

So far we have given various arguments towards concluding that the ACG-cubic metric is isometric to the Bergmann metric. In appendix A, we provide an explicit co-ordinate transformation between the polar co-ordinates and the ring-like Darboux co-ordinates.

4.2 AdS_5

With a base space that is Kähler-Einstein, we can get a solution to the Gauntlett-Gutowski theory i.e. a solution in the time-like class to minimal gauged supergravity by choosing $f = 1$ and the one-form ω as (section 3.2 of [10],)

$$\begin{aligned} \omega_{AdS_5} &= \sqrt{p}\,[(\xi+\eta)\;dt + (1+\xi\,\eta)\;dz] \\ &= \sqrt{p}\,\Omega^{(1)} \end{aligned} \tag{4.10}$$

where $\Omega^{(1)}$ is the one-form that is natural to the Kähler structure given in (2.7). We can check by an explicit computation that the metric,

$$ds^2 = -[d\tau + \omega_{AdS_5}]^2 \;+\; ds^2_{\mathrm{ACG-cubic}}. \tag{4.11}$$

is a maximally symmetric metric with Ricci scalar $-5\,p$. To confirm that this is indeed AdS_5 space, we can invoke the following theorem proved by Gutowski and Reall in [17]: *the only maximally symmetric solution in the time-like class to minimal gauged supergravity is* AdS_5 *and the base space is locally isometric to the Bergmann manifold.*

To reiterate, in this section, we have obtained the Bergmann metric and AdS_5 in ring-like co-ordinates in the same co-ordinate region as the susic flat ring.

5. The Kähler structure of susic flat solutions

In this section, we will first co-ordinate transform known $\mathrm{U}(1)^2$ invariant asymptotically flat susic solutions to the ring-like Darboux co-ordinates.

For the supersymmetric flat-ring solution [5], the f and ω are given in C-metric co-ordinates by

$$\frac{1}{f} = 1 + \frac{Q-q^2}{2}(x-y) - \frac{q^2}{4}\left(x^2-y^2\right) \tag{5.1}$$

and $\omega = \omega_\phi\; d\phi + \omega_\psi\; d\psi$, with

$$\omega_\phi = -\frac{q}{8}(1-x^2)[3Q - q^2(3+x+y)] \tag{5.2}$$

$$\omega_\psi = \frac{3\,q}{2}(1+y) + \frac{q}{8}(1-y^2)[3Q - q^2(3+x+y)]. \tag{5.3}$$

[6]We thank Toby Wiseman for suggesting this test for a homogeneous metric.

where Q is the charge of the ring and q it's dipole charge. Using the co-ordinate transformations (3.15), the flat-ring solution takes the following form in the ring-like Darboux co-ordinates.

$$\frac{1}{f} = 1 + \frac{Q - q^2}{\Lambda} - 4\, q^2\, \frac{\xi + \eta}{\Lambda^2} \tag{5.4}$$

where Λ is given in (3.16). The one-form ω is now $\omega = \omega_t\, dt + \omega_z\, dz$, with

$$\omega_t = \frac{3\,q}{4} + \frac{3\,q}{4\,\Lambda}(4\xi + 4\eta - 1) + \frac{3\,q\,(Q - q^2)}{\Lambda^2}(\xi + \eta) - \frac{8\,q^3}{\Lambda^3}(\xi + \eta)^2 \tag{5.5}$$

$$\omega_z = -\frac{3\,q}{4} - \frac{3\,q}{4\,\Lambda}(4\xi + 4\eta - 1) + \frac{3\,q\,(Q - q^2)}{\Lambda^2}(1 + \xi\eta) - \frac{8\,q^3}{\Lambda^3}(\xi + \eta)(1 + \xi\,\eta). \tag{5.6}$$

It is possible to describe a supersymmetric black hole in the background of the supersymmetric black-ring by a simple modification of the $\frac{1}{f}$ of the solution [7, 8, 12, 13]. The black hole by itself has a $\mathrm{SU}(2) \times \mathrm{U}(1)$ isometry and doesn't disturb the $\mathrm{U}(1)^2$ isometry of the ring on its own. The modification involves adding the following harmonic function, harmonic with respect to the Laplacian on R^4,

$$\frac{1}{f} = 1 - \frac{Q_{\mathrm{BH}}}{16}\left(\frac{x - y}{x + y}\right) + \frac{Q - q^2}{2}(x - y) - \frac{q^2}{4}\left(x^2 - y^2\right), \tag{5.7}$$

where Q_{BH} is the charge of the black hole. The one-form ω for this system is given by

$$\omega_\phi = -\frac{q}{8}(1 - x^2)\left[3Q - q^2(3 + x + y) - \frac{9\,Q_{\mathrm{BH}}}{4}\frac{1}{x + y} + 2K\frac{1}{(x + y)^2}\right] \tag{5.8}$$

$$\omega_\psi = \frac{3\,q}{2}(1 + y) + \frac{q}{8}(1 - y^2)\left[3Q - q^2(3 + x + y) - \frac{9\,Q_{\mathrm{BH}}}{4}\frac{1}{x + y} + 2K\frac{1}{(x + y)^2}\right], \tag{5.9}$$

where K is a constant proportional to the rotation of the black hole. Co-ordinate transforming to the ring-like Darboux co-ordinates, we have,

$$\frac{1}{f} = 1 - \frac{Q_{\mathrm{BH}}}{64}\frac{1}{\xi + \eta} + \frac{Q - q^2}{\Lambda} - 4\,q^2\,\frac{\xi + \eta}{\Lambda^2} \tag{5.10}$$

and the one-form ω is:

$$\omega_t = \omega_{t(0)} - \frac{9\,q\,Q_{\mathrm{BH}}}{32\,\Lambda} + \frac{K\,q}{32\,(\xi + \eta)} \tag{5.11}$$

$$\omega_z = \omega_{z(0)} - \frac{9\,q\,Q_{\mathrm{BH}}}{32\,\Lambda}\frac{1 + \xi\eta}{\xi + \eta} + \frac{K\,q(1 + \xi\eta)}{32\,(\xi + \eta)^2} \tag{5.12}$$

where $\omega_{t(0)}$ and $\omega_{z(0)}$ are the expressions for pure black ring (i.e. without the black hole) as given in formulae (5.5) and (5.6).

We first note that the ω for the supersymmetric flat ring (5.5), (5.6) can be written in the following suggestive manner:[7]

$$\begin{aligned}\omega &= \rho(\xi,\eta)\,\frac{dt - dz}{2} + \frac{\kappa(\xi,\eta)}{\Lambda^2}\,\left[(\xi + \eta)\,dt + (1 + \xi\eta)\,dz\right] \\ &= \rho(\xi,\eta)\,d\psi + \frac{\kappa(\xi,\eta)}{\Lambda^2}\,\Omega^{(1)},\end{aligned} \tag{5.13}$$

[7] It is perhaps not irrelevant to note that the ω for the non-supersymmetric flat ring of [1] also is of the form (5.13) with $\kappa(\xi,\eta) = 0$; of course, the stationary form of the metric is not as simple as (1.1).

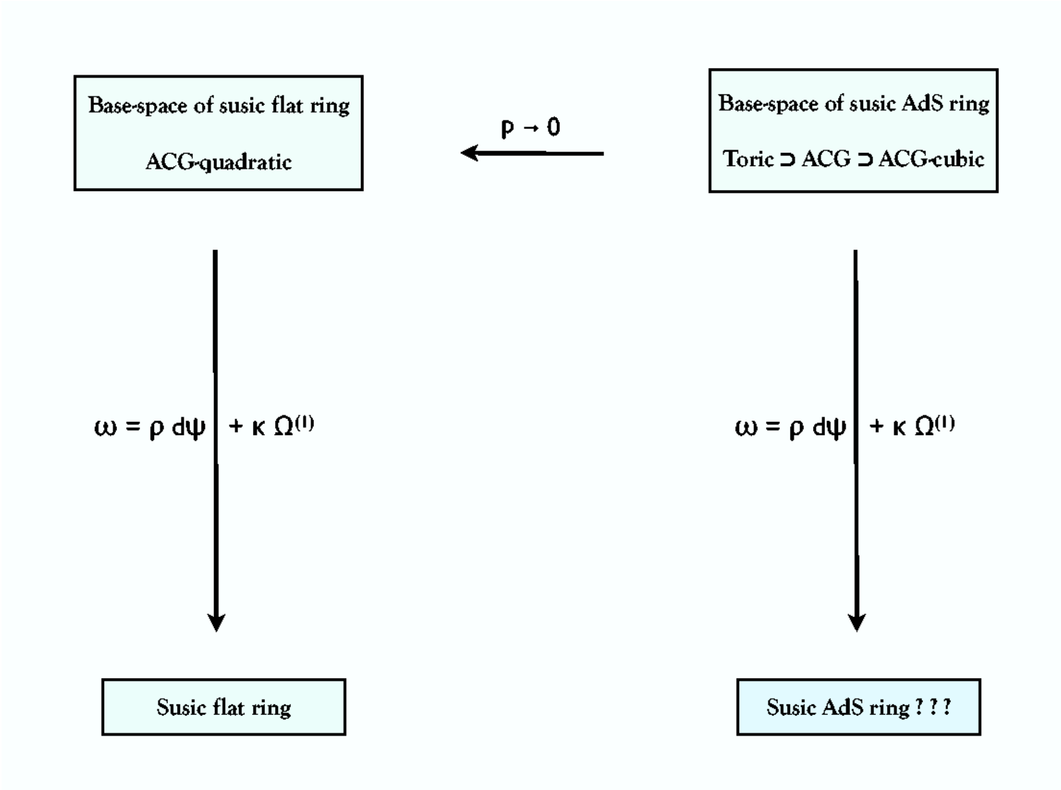

Figure 2: A schematic representation of our approach to obtaining a supersymmetric AdS ring.

where $\Omega^{(1)}$ is the one-form natural to the Kähler structure given in (2.7). For the flat ring, the functions ρ and κ are ρ_0 and κ_0 given by,

$$\rho_0(\xi,\eta) = \frac{3q}{2}\left(1+\frac{4\xi+4\eta-1}{\Lambda}\right) \tag{5.14}$$

$$\kappa_0(\xi,\eta) = 3q(Q-q^2) - 8\,q^3\frac{\xi+\eta}{\Lambda}. \tag{5.15}$$

The ω for the black hole in the background of the flat ring also takes the above form (5.13), for which the functions ρ and κ

$$\rho(\xi,\eta)_{\mathrm{BH\,in\,a\,Ring}} = \rho_0(\xi,\eta) \tag{5.16}$$

$$\kappa(\xi,\eta)_{\mathrm{BH\,in\,a\,Ring}} = \kappa_0(\xi,\eta) - \frac{9\,q\,Q_{\mathrm{BH}}}{32}\frac{\Lambda}{\xi+\eta} + \frac{Kq}{32}\left(\frac{\Lambda}{\xi+\eta}\right)^2. \tag{5.17}$$

This form for ω (5.13), (1.4) seems to encode in it the fact that there is a ring (through the $d\psi$ term), the underlying Kähler structure (through the term $\Omega^{(1)}$) and the $\mathrm{U}(1)^2$ invariance (because all known $\mathrm{U}(1)^2$ invariant solutions take this form.)

6. An ansatz for a supersymmetric AdS ring?

Our approach to a susic AdS ring is summarized schematically in figure 2.

6.1 An ansatz for the AdS ring Kähler metric

As summarized in the box below, we have many classes of $\mathrm{U}(1)^2$ invariant Kähler metrics from which to pick an ansatz for the AdS ring.

$$\text{Weakly self-dual ortho-toric} \;\subset\; \text{ACG ortho-toric} \;\subset\; \text{Toric Kähler metrics.}$$

The first class, i.e. weakly-self-dual ortho-toric, has the most explicit classification; it is completely determined by the quartic polynomials (2.9). To decide if we can make an ansatz for the AdS ring with a weakly self-dual ortho-toric Kähler metric, we checked if the Gutowski-Reall black hole [17, 18] Kähler metric is weakly self-dual and found that it *isn't*. The details of this can be found in the appendix. Hence, we rule out a weakly-self-dual ansatz.

The second class i.e. the ACG metrics (2.4) admit a less explicit classification; there are two arbitrary functions in the game. The functions are functions of one variable, which is a simplifying feature, giving total derivatives rather than partial derivatives for the connections and curvatures. The third class i.e the toric Kähler metrics (2.1) admit an even less explicit classification; the metric involves functions of two variables. One could simplify things a bit by working with a single function of two variables, the symplectic potential. It remains to be seen which of the two classes of metrics will contain the AdS ring. We will report further progress in this direction in future work [26]. Both classes of metrics include the two limiting cases viz. the asymptotic ACG-cubic metric and the $p \to 0$ limit, i.e. the ACG-quadratic metric.

6.2 An ansatz for ω

The form (5.13) for the one-form ω seems to capture some crucial features that we desire for a supersymmetric AdS ring, viz. Kähler structure, $\mathrm{U}(1)^2$ invariance etc. Since the asymptotic AdS_5 metric and the full AdS ring Kähler metric ansatz share the same Kähler structure (i.e. both Ω and $\Omega^{(1)}$), we can happily make the same ansatz (5.13) for the ω of the supersymmetric AdS ring. We also have the following information for the ansatz functions ρ and κ. At asymptotic infinity, $\rho \to 0, \kappa \to \sqrt{p}\ \Lambda^2$. And as $p \to 0$, $\rho \to \rho_0, \kappa \to \kappa_0$. The observation that $\kappa(\xi, \eta)$ is a function of the combination $\frac{\xi+\eta}{\Lambda}$ should also be useful.

We now need to plug these Ansätze into the Gauntlett-Gutowski theory. The Gauntlett-Gutowski theory of supersymmetric solutions works somewhat differently to the theory of supersymmetric solutions to gauged supergravity. First, the $\frac{1}{f}$ of the solution gets completely determined by the Ricci scalar of the base-space Kähler metric and second, the G^+ of the solution gets completely determined by the traceless Ricci form of the base-space Kähler metric. These two facts constrains our Ansätze above. Finally the G^- of the solution is determined by a first-order p.d.e which itself is determined fully by the base-space Kähler metric.[8] We will leave this for future work [26].

[8]We would also need to take care of the addition to the Gauntlet-Gutowski theory given in [27], which ensures that the $d\omega$ thus determined is indeed closed.

7. Conclusion

Starting from the requirements of $\mathrm{U}(1)^2$ isometry and a Kähler base, which are natural to supersymmetric AdS ring solutions, enabled us to get some insight into the supersymmetric solutions. Coordinates (related to the) moment maps of the two U(1)'s turn out to be "ring-like." We have described AdS_5 and other solutions in these ring-like co-ordinates and have given an Ansätz for the supersymmetric AdS ring solution and a strategy for obtaining it (figure 2). Since [24, 25] casts doubt on the existence of supersymmetric AdS rings, our ansatz provides a context for addressing this question further. Recently a non-supersymmetric black ring solution in de-Sitter space was constructed [28]. This is encouraging because it suggests that non-supersymmetric black rings can exist in AdS. The toric almost-Kähler[9] metrics of (2.1) provide a metric ansatz to start with for non-supersymmetric AdS rings.

Acknowledgments

We would like to thank Justin David, Edward Teo, Martin O' Loughlin, Harvey Reall, Boris Pioline, Koushik Ray and Toby Wiseman for discussions.

A. The Fubini-Study metric as an ACG-cubic metric

The ACG-cubic metric (4.2) has two parameters, a and p while the Bergmann metric has only one parameter p. To conclude that the two metrics are isometric, we need to show that a is a co-ordinate artifact or even better display an explicit co-ordinate transformation that settles this issue. We do this in the context of Fubini-Study metric on $\mathbf{CP}^2$, which is the compact version of the Bergmann metric. It can be simply obtained by replacing $(1-r^2) \to (1+r^2)$ in (4.8). More precisely, it is the Kähler metric obtained from the Kähler potential,

$$K = \frac{2}{p} \log(1 + |z^1|^2 + |z^2|^2), \tag{A.1}$$

which takes the explicit form,

$$ds^2_{\mathrm{FS}} = \frac{4}{p}\left[\frac{dr^2}{(1+r^2)^2} + \frac{r^2}{4(1+r^2)}(d\theta^2 + \sin\theta^2 d\phi^2) + \frac{r^2}{4(1+r^2)^2}(d\psi + \cos\theta\, d\phi)^2\right], \tag{A.2}$$

in the complex structure

$$z^1 = r\,\cos\frac{\theta}{2}\, e^{\frac{i(\phi+\psi)}{2}}, \qquad z^2 = r\,\sin\frac{\theta}{2}\, e^{\frac{i(\phi-\psi)}{2}}. \tag{A.3}$$

The Fubini-Study metric is also a Kähler-Einstein metric with constant holomorphic sectional curvature. In this appendix, we will show that the Fubini-Study metric can be described as an ACG-cubic metric. First consider the following general ACG-cubic metric,

$$F(x) = G(x) = -p\,(x-b_1)(x-b_2)(x-b_3). \tag{A.4}$$

[9] An almost Kähler manifold is not a complex manifold and hence not Kähler.

Either by direct computation or using the results of [11], one can ascertain that this is a Kähler-Einstein metric of Ricci scalar 6 p. We can also directly compute and verify that the above ACG-cubic metric has constant holomorphic sectional curvature. Under the co-ordinate transformation

$$x_1 = \xi + \eta, \qquad\qquad x_2 = \xi\,\eta, \tag{A.5}$$

$$\xi = \frac{1}{2}\left(x_1 + \sqrt{x_1^2 - 4x_2}\right), \qquad\qquad \eta = \frac{1}{2}\left(x_1 - \sqrt{x_1^2 - 4x_2}\right) \tag{A.6}$$

to the moment map co-ordinates, the ACG-cubic metric takes a certain form which we won't reproduce here. But more importantly, this metric can be derived from the following symplectic potential,

$$\begin{aligned} G_S(x_1, x_2) = &-\frac{x_2 - b_1 x_1 + b_1^2}{p\,(b_1 - b_2)(b_3 - b_1)} \log\left[-\frac{2(x_2 - b_1 x_1 + b_1^2)}{p\,(b_1 - b_2)(b_3 - b_1)}\right] \\ &-\frac{x_2 - b_2 x_1 + b_2^2}{p\,(b_2 - b_3)(b_1 - b_2)} \log\left[-\frac{2(x_2 - b_2 x_1 + b_2^2)}{p\,(b_2 - b_3)(b_1 - b_2)}\right] \\ &-\frac{x_2 - b_3 x_1 + b_3^2}{p\,(b_3 - b_1)(b_2 - b_3)} \log\left[-\frac{2(x_2 - b_3 x_1 + b_3^2)}{p\,(b_3 - b_1)(b_2 - b_3)}\right] \end{aligned} \tag{A.7}$$

using the formula (2.1). We will not go into the details of how we obtained (A.7) from the ACG data (A.4). For the complete theory of symplectic potentials for the ACG metrics and every other consideration in this appendix, we refer the reader to [29]. From the work of Guillemin [14], it is known that the above symplectic potential (A.7) encodes in it the co-ordinate singularities of a toric manifold whose moment polytope is the convex set enclosed within the lines,

$$l_i \equiv x_2 - b_i\,x_1 + b_i^2 = 0, \qquad i = 1, 2, 3. \tag{A.8}$$

In the $x_1 - x_2$ plane, $l_i = 0$ is a line with slope b_i and x_2-intercept equalling $-b_i^2$. The intersection of the lines are:

$$l_1 \cap l_2 : (b_1 + b_2, b_1\,b_2), \qquad l_2 \cap l_3 : (b_2 + b_3, b_2\,b_3), \qquad l_3 \cap l_1 : (b_3 + b_1, b_3\,b_1). \tag{A.9}$$

We thus have that the moment polytope of the ACG-cubic metric (A.4) is a triangle with vertices (A.9) and edges (A.8).

At this stage, we have used up all the local properties of the Fubini-Study metric, namely Kähler-Einstein and constant holomorphic sectional curvature, and we still have three undetermined constants in the ACG-cubic metric (A.3) viz. b_1, b_2, b_3. The Fubini-Study metric (A.2) has no free parameters apart from the Ricci-scalar ($\sim p$). We will have to use the global features of the Fubini-Study metric and $\mathbf{CP}^2$ to fix the parameters b_i. Global considerations specify the size and shape of the triangle which in turn fixed the b_i's. Before we do this, we give the co-ordinate transformation between the Darboux co-ordinates and the polar co-ordinates:

$$\begin{aligned} r^2 &= \frac{b_1\,(b_2-b_3) - b_2 b_3 + (\xi+\eta)\,b_3 - \xi\eta}{b_3^2 - (\xi+\eta) b_3 + \xi\eta}, \\ \tan^2\theta/2 &= \frac{(b_2^2 - (\xi+\eta)\,b_2 + \xi\eta)(b_3-b_1)}{(b_1^2 - (\xi+\eta)\,b_1 + \xi\eta)(b_2-b_3)}, \\ \phi &= -\frac{p}{2}\,(b_1-b_2)\,(t+b_3 z)\,, \\ \psi &= \frac{p}{2}\,[(2b_3-b_1-b_2)t - (2b_1b_2 - b_1b_3 - b_2b_3)z]\,, \end{aligned} \tag{A.10}$$

and the inverse,

$$\begin{aligned} \xi+\eta &= \frac{2(b_1+b_2) + (b_1+b_2+2b_3)r^2 - (b_1-b_2)r^2\cos\theta}{2(1+r^2)}, \\ \xi\,\eta &= \frac{2b_1b_2 + (b_1+b_2)b_3\,r^2 - (b_1-b_2)b_3\,r^2\cos\theta}{2(1+r^2)}, \\ t &= \frac{(2b_1b_2 - b_2b_3 - b_3b_1)\phi + (b_2b_3 - b_3b_1)\psi}{p\,(b_1-b_2)\,(b_2-b_3)\,(b_3-b_1)}, \\ z &= \frac{(2b_3-b_1-b_2)\phi + (b_1-b_2)\psi}{p\,(b_1-b_2)\,(b_2-b_3)\,(b_3-b_1)}. \end{aligned} \tag{A.11}$$

To get the above, one first goes from symplectic co-ordinates to complex co-ordinates by a Legendre transform [14, 15]. The Legendre transform also provides the Kähler potential. Then one matches with (A.1) and (A.3). It is clear, especially from (A.10), that the parameters b_i appear only in the co-ordinate transformation, and hence do not carry any local co-ordinate invariant information.

Global considerations. Now, we can fix the parameters b_i by the following two facts that carry global information of the Fubini-Study metric and $\mathbf{CP}^2$.

(i) The moment polytope of $\mathbf{CP}^2$ is a right-angled isosceles triangle.

(ii) The volume of $\mathbf{CP}^2$ is fixed once one specifies the Ricci-scalar. For (A.2), it is $\frac{8\pi^2}{p^2}$

The easiest way of seeing (i) is to note that in the GLSM description of $\mathbf{CP}^2$, there is one D-term constraint, $|\phi_1|^2 + |\phi_2|^2| + |\phi_3|^2 = \frac{2}{p}$, which is solved by

$$|\phi_1|^2 = x_1, \qquad |\phi_2|^2 = x_2, \qquad |\phi_3|^2 = \frac{2}{p} - x_1 - x_2. \tag{A.12}$$

The moment polytope is then given by the region

$$|\phi_1|^2 > 0, \qquad |\phi_2|^2 > 0, \qquad |\phi_3|^2 > 0, \tag{A.13}$$

which is clearly a right-angled isosceles triangle of area $\frac{2}{p^2}$.

Let us choose the right angle to be at the intersection of $l_1 = 0$ and $l_2 = 0$. We then have the following (two independent) constraints:

$$b_1\,b_2 = -1, \qquad \frac{b_1 - b_3}{1 + b_1\,b_3} = 1, \qquad \frac{b_3 - b_2}{1 + b_3\,b_2} = 1. \tag{A.14}$$

This allows us to fix two of the parameters in terms of one free parameter, say b_3:

$$b_1 = \frac{1+b_3}{1-b_3}, \qquad b_2 = -\frac{1-b_3}{1+b_3}. \tag{A.15}$$

Now we will impose the volume constraint. In the symplectic co-ordinates, by virtue of the fact that the four-metric is of a block-diagonal form (2.1) with two 2×2 matrices which are inverses of each other, the determinant of the metric is just 1. Hence, the volume of the manifold is just the product of the Euclidean volumes of the moment polytope and the angular torus. From (A.9) and from (A.15), we can compute the Euclidean volume of the moment polytope in terms of b_3. To compute the volume of the angular torus, we need the co-ordinate transformation (A.11). Using the ranges of the angles $0 < \phi < 2\pi,\ 0 < \psi < 4\pi$ and the second two equations in (A.11), we can compute the Euclidean volume of the angular torus. Putting everything together, we have

$$\mathrm{Vol}_{\mathrm{ACG-cubic}} = \frac{4\pi^2}{p^2}\frac{(2-b_3)(1+2b_3)}{1+b_3^2}. \tag{A.16}$$

Now, requiring $\mathrm{Vol}_{\mathrm{ACG-cubic}} = \mathrm{Vol}_{\mathbf{CP}^2} = \frac{8\pi^2}{p^2}$, we get a quadratic equation for b_3 with two roots $b_3 = 0, \frac{3}{4}$. The two roots suggest that we have two different ways of realizing the Fubini-Study metric as an ortho-toric ACG metric. One of the solutions, $b_3 = 0$ gives us co-ordinate singularities which are "ring-like." Using (A.15), we then have,

$$b_1 = 1, \qquad b_2 = -1, \qquad b_3 = 0, \tag{A.17}$$

so that the polynomials occuring in the ACG-cubic metric are

$$F(z) \;=\; G(z) \;=\; (1-z^2)\,(p\,z). \tag{A.18}$$

The explicit co-ordinate transformation is now much simpler:

$$\begin{aligned} r^2 &= -\frac{1+\xi\eta}{\xi\eta}, \qquad \tan^2\theta/2 = \frac{1+\xi\eta+(\xi+\eta)}{1+\xi\eta-(\xi+\eta)}, \qquad \phi = -p\,t, \qquad \psi = p\,z; \\ \xi+\eta &= -\frac{r^2\cos\theta}{1+r^2}, \qquad \xi\,\eta = -\frac{1}{1+r^2}, \qquad t = -\frac{\phi}{p}, \qquad z = \frac{\psi}{p}. \end{aligned} \tag{A.19}$$

In this appendix, we have been able to show the explicit isometry between the Fubini-Study metric and a compact ACG-cubic metric by deriving an explicit co-ordinate transformation between the polar co-ordinates of the form and the "ring-like" co-ordinates of the latter. A similar co-ordinate transformation relates the Bergmann metric to the ACG-cubic metric; the co-efficient of the cubic has the opposite sign reflecting the sign of the scalar curvature. However, one free parameter remains as Bergmann space is non-compact.

B. The Gutowski-Reall black hole Kähler metric and weak self-duality

In the recent past, many researchers have constructed various supersymmetric AdS_5 black holes [17 − 19, 22, 23], but we will only need to consider the first ones and the simplest

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of them, the Gutowski-Reall black hole. The Kähler metric of the Gutowski-Reall black holes [17, 18] is

$$ds^2 = dr^2 + a^2((\sigma_L^1)^2 + (\sigma_L^2)^2) + (2aa')^2(\sigma_L^3)^2, \tag{B.1}$$

where $a(r)$ is

$$a(r) = \sqrt{\frac{\alpha}{4} + \frac{1}{p}} \sinh \frac{\sqrt{p}\, r}{2}, \tag{B.2}$$

with α is a constant and σ_L^i are the right-invariant one-forms on SU(2), which can be expressed in terms of the Euler angles (θ, ϕ, ψ) as

$$\begin{aligned}
\sigma_L^1 &= \sin\phi \; d\theta - \cos\phi \; \sin\theta \; d\psi, \\
\sigma_L^2 &= \cos\phi \; d\theta + \sin\phi \; \sin\theta \; d\psi, \\
\sigma_L^3 &= d\phi + \cos\theta \; d\psi.
\end{aligned} \tag{B.3}$$

This metric (B.1) is Kähler with the Kähler form

$$\Omega = \; d(a^2\sigma_L^3) \; = \; 2aa' \; dr \wedge \sigma_L^3 \; - \; a^2 \; \sigma_L^1 \wedge \sigma_L^2 \tag{B.4}$$

In the orientation with $\mathrm{Vol} = 2a^3 a' \sin\theta \, dr \wedge d\theta \wedge d\phi \wedge d\psi$, the Kähler form (B.4) is self-dual. We can check for the weak-self-duality property of a given Kähler metric in atleast two ways [11].

(i) The defining property of a weakly self-dual Kähler metric is that the anti-self-dual Weyl tensor is harmonic, i.e. $\delta W^- = 0$ where δ acts on W^- as on a two-form with values in anti-self-dual two-forms. On the other hand, the Weyl tensor on a Riemannian four-manifold is linked with the Cotton-York tensor, $\delta W^\pm \; = \; C^\pm$. Hence, vanishing of the anti-self-dual Cotton-York tensor implies the weak self-duality of a given Kähler metric (see definition 2 in [11].)

The Cotton-York tensor is defined in [11], $C_{X,Y}(Z) := (\nabla_Y h)(X,Z) - (\nabla_X h)(Y,Z)$, where h is the normalized Ricci tensor given by $h_{\mu\nu} \; = \; \frac{1}{2}R_{\mu\nu} - \frac{R}{12} g_{\mu\nu}$. Define it's components by $C_{X,Y}(Z) := C_{kij} Z^k X^i Y^j$ and we get

$$C_{kij} = \frac{1}{2}\left(\nabla_j R_{ik} - \nabla_i R_{jk}\right) - \frac{1}{12}\left(g_{ik}\nabla_j R - g_{jk}\nabla_i R\right). \tag{B.5}$$

The indices i, j are anti-symmetric and one is supposed to think of the Cotton-York tensor as a co-vector-(the index k)-valued two-form, $C_r^{(2)}, C_\theta^{(2)}, C_\phi^{(2)}, C_\psi^{(2)}$. Plugging in (B.1) into (B.5), we get

$$\begin{aligned}
C_r^{(2)} &= 0, \\
C_\theta^{(2)} &= -\frac{\alpha\sqrt{p^3}}{48} \coth\frac{\sqrt{p}\, r}{2} \; dr \wedge d\theta + \frac{\alpha\, p^3}{256}\left(1 + \frac{\alpha\, p}{4}\right)\cosh^2\frac{\sqrt{p}\, r}{2} \sin\theta \; d\phi \wedge d\psi, \\
C_\phi^{(2)} &= \frac{\alpha\sqrt{p^7}}{384}\left(1 + \frac{\alpha\, p}{4}\right)\cosh^2\frac{\sqrt{p}\, r}{2} \; \coth\frac{\sqrt{p}\, r}{2} \; dr \wedge \sigma_L^3 \\
&\quad + \frac{\alpha\sqrt{p^3}}{128}\left(1 + \frac{\alpha\, p}{4}\right)\cosh^2\frac{\sqrt{p}\, r}{2} \; \sigma_L^1 \wedge \sigma_L^2, \\
C_\psi^{(2)} &= \ldots
\end{aligned} \tag{B.6}$$

Computing the anti-self-dual parts of the above two-forms gives us,

$$\begin{aligned}
C_r^- &= 0, \\
C_\theta^- &= -\frac{\alpha\sqrt{p^3}}{24}\coth\frac{\sqrt{p}\,r}{2}\,dr\wedge d\theta + \frac{\alpha\,p^3}{384}\left(1+\frac{\alpha\,p}{4}\right)\cosh^2\frac{\sqrt{p}\,r}{2}\sin\theta\;d\phi\wedge d\psi, \\
C_\phi^- &= \frac{\alpha\sqrt{p^7}}{192}\left(1+\frac{\alpha\,p}{4}\right)\cosh^2\frac{\sqrt{p}\,r}{2}\;\coth\frac{\sqrt{p}\,r}{2}\;dr\wedge\sigma_L^3 \\
&\quad+\frac{\alpha\,p^3}{192}\left(1+\frac{\alpha\,p}{4}\right)\cosh^2\frac{\sqrt{p}\,r}{2}\;\sigma_L^1\wedge\sigma_L^2, \\
C_\psi^- &= \ldots
\end{aligned} \tag{B.7}$$

We thus find that the anti-self-dual part of the Cotton-York tensor is non-vanishing for the black hole Kähler metric, thus making it *not* weakly self-dual. Note that for $\alpha = 0$, the Kähler metric is weakly self-dual, which is as it should be because it corresponds to the Bergmann Kähler metric.

(ii) On a complex two-dimensional Kähler manifold (g, J, Ω_J) with orientation chosen so that the Kähler form, Ω_J, is self-dual, a property of the traceless Ricci form, ρ_0, is that it is anti-self-dual. Any anti-self-dual two-form can always be written as a functional multiple of an anti-self-dual Kähler form of some other (needn't be integrable) almost-complex structure, $\rho_0 = \lambda\,\Omega_I$, where (g, I, Ω_I) is an almost-hermitian structure. One of the many defining properties of a weakly self-dual Kähler metric is that $(\frac{g}{\lambda^2}, I, \frac{\Omega_I}{\lambda^2})$ is Kähler (see lemma 2 and lemma 4 of [11].)

For the Kähler metric (B.1), the traceless Ricci form is

$$\rho_0 = b\;dr\wedge\sigma_L^3 + \frac{b\,a}{2a'}\;\sigma_L^1\wedge\sigma_L^2, \tag{B.8}$$

where $b(r)$ is

$$b(r) = -\frac{a'''a^2 + 3a''a'a - 4a'^3 + a'}{a}. \tag{B.9}$$

This ρ_0 can be written as $\lambda\;\Omega_I$ with

$$\Omega_I = \;2aa'\;dr\wedge\sigma_L^3\;+\;a^2\;\sigma_L^1\wedge\sigma_L^2 \tag{B.10}$$

and

$$\lambda(r) = \frac{b}{2aa'}. \tag{B.11}$$

It is easy to see that Ω_I is anti-self-dual and one can construct the almost complex structure $I \sim g^{-1}\;\Omega_I$ and verify $I^2 = -Id$. For the Kähler metric (B.1) to be weakly self-dual,

$$d\,\left(\frac{\Omega_I}{\lambda^2}\right) \;=\; 0, \tag{B.12}$$

which amounts to the following fourth order non-linear o.d.e:

$$\left(\frac{a'^2a^6}{b^2}\right)' + \frac{2a'^3a^3}{b^2} = 0. \tag{B.13}$$

But, the black hole Kähler metric (B.2) does *not* satisfy the above o.d.e, hence *not* weakly self-dual.

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